OPTIMUM CORRECTION UNDER ACTIVE DISTURBANCES

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Certain model problems of optimal correction of motion of a system subject not only to controlling forces, but also to uncontrolled forces (disturbances), are investigated. The measurement errors are not taken into account. It is assumed that the disturbances are active and have the most unfavorable effect from the controller's standpoint. The problems are solved in minimax (games-theoretical) formulation, and the resulting optimal solutions are of guaranteeing character. As compared with the conventional statistical approach, the minimax method has certain drawbacks (the minimax strategy is sometimes over-cautious) as well as several advantages: 1) knowledge of the probabilistic characteristics of the disturbances, which are often unknown, is not required; 2) the minimax approach is applicable even in cases where the disturbances are produced by the active apponent; 3) the result afforded by the minimax approach is more reliable.

Several variants of correction problems in which both the controlling forces and the disturbances can be either pulsed or bounded are investigated. The solutions obtained are used to draw certain qualitative conclusions concerning the relative effectiveness of continuous and pulse correction.

1. Formulation of the problem. Let the motion of a controlled system be described by the differential equations and initial conditions and restrictions

$$dx' / dt' = y', \qquad dy' / dt' = u' + v', \qquad x'(0) = y'(0) = 0$$
$$\int_0^T |u'(t')|\, dt' \leqslant p, \qquad \int_0^T |v'(t')|\, dt' \leqslant q \tag{1.1}$$

Here t' is the time, x' is the generalized coordinate vector, y' is the velocity vector, u' is the controlling force vector, v' is the disturbing force vector, T is the prescribed time of termination of the process, and p and q are the prescribed total magnitudes of the correction and disturbance force pulses. All of the vectors in (1.1) are of arbitrary (but the same) dimensionality.

Eqs. (1.1) can be regarded as equations in variations relative to some unperturbed nominal system trajectory.

We are required to choose the control $u'(t')$ over the interval $[0, T]$ in such a way as to minimize the length of the generalized coordinate vector at the end of the process, i.e. to minimize $|x'(T)|$. We assume that the disturbances $v'(t')$ are chosen on the basis of the maximization condition for the functional $|x'(T)|$. In other words, we consider the problem in minimax (games-theoretical) formulation. We assume that both of the controlling players can measure the present coordinates x' and velocities y' of the system exactly at each instant.

Let us convert to dimensionless variables which we denote by unprimed letters,

$$t' = Tt, \quad x' = qTx, \quad y' = qy, \quad u' = qT^{-1}u, \quad v' = qT^{-1}v, \quad p = qk \quad (1.2)$$

Expressed in terms of variables (1.2), relations (1.1) become

$$dx/dt = y, \qquad dy/dt = u + v, \qquad x(0) = y(0) = 0$$

$$\int_0^1 |u(t)|\,dt \leqslant k, \qquad \int_0^1 |v(t)|\,dt \leqslant 1 \qquad (1.3)$$

The control $u(t)$ must be chosen from the minimization condition for the functional $|x(1)|$; the disturbance $v(t)$ must be chosen from the maximization condition for this functional. In addition to conditions (1.3), we also impose the following restrictions on the choice of the functions u and v: these functions are either bounded in absolute value, or they are of a single-pulse character. In order to satisfy integral restrictions (1.3), we require fulfillment of the inequalities

$$|u(t)| \leqslant k, \quad |v(t)| \leqslant 1 \quad \text{for } 0 \leqslant t \leqslant 1$$

in the former case, and set

$$u(t) = U\delta(t - \tau), \qquad v(t) = V\delta(t - \theta) \qquad (|U| \leqslant k, \ |V| \leqslant 1)$$

in the latter case.

Here δ is a delta function, τ and θ are arbitrary instants in the interval $[0, 1]$, and U, V are constant vectors. We shall solve the problem for various combinations of these restrictions and without them.

2. Continuous correction of a continuous disturbance. Let the controlling functions in (1.3) be subject to the restrictions $|u(t)| \leq k$, $|v(t)| \leq 1$ for $0 \leq t \leq 1$. The correction problem then reduces to a differential games problem [1] for system (1.3) with the functional $|x(1)|$.

This problem can be solved readily by elementary argument, without resorting to differential games theory. First, let $k \geq 1$. Since both players have complete information on the motion of the system, the controlling player (controller) can ensure fulfillment of Eq. $u(t) = -v(t)$ with any degree of accuracy for all t.

The functional of the problem in this case is $J_1(k) = |x(1)| = 0$, where the subscript denotes the number of the case under consideration. If $k < 1$, then, clearly, it is best to make the control of maximal magnitude and to direct it opposite to the disturbance. The disturbance in this case is of maximal magnitude; its direction is arbitrary, but constant (there is no need to vary its direction in view of the availability of complete information).

Hence, $u = -ke$, $v = e$ for all t, where e is an arbitrary unit vector. The functional in this case is given by $J_1(k) = (1 - k)/2$. Thus, the functional $|x(1)|$ for optimal correction in the case just considered is

$$J_1 = (1 - k)/2 \quad \text{for } 0 \leqslant k \leqslant 1, \qquad J_1 = 0 \quad \text{for } k \geqslant 1 \qquad (2.1)$$

3. Pulse correction of a continuous disturbance. The disturbance is restricted by the condition $|v(t)| \leq 1$ for $0 \leq t \leq 1$ as above; the correcting control is of the form $u(t) = U\delta(t - \tau)$, where the vector U at the correction instant τ is chosen from the domains $|U| \leq k$, $0 \leq \tau \leq 1$. The functional J_2 in this case is given by Eq.

$$J_2 = \min_{\tau} \max_{v[0,\tau]} \min_{U} \max_{v[\tau,1]} |x(1)| \qquad (3.1)$$

The sequence of extrema in this case conforms to our hypothesis whereby both sides are completely informed. We shall compute the extrema in (3.1) by working backwards. To determine the last maximum in (3.1) we must solve the problem of optimal control of the system

$$dx/dt = y, \qquad dy/dt = v, \qquad |v| \leqslant 1 \qquad (3.2)$$

in the interval $[\tau, 1]$ under the initial conditions and with the functional

$$x(\tau) = x^+, \quad y(\tau) = y^+, \quad x_0(1) = -x^2(1) \to \min$$
$$dx_0 / dt = -2(x, y), \qquad x_0(\tau) = -(x^+)^2 \tag{3.3}$$

Here $x^+ = x(\tau + 0)$, $y^+ = y(\tau + 0)$ are the values of the coordinates and velocities immediately following pulse correction; $x_0 = -x^2$ is an ancillary phase coordinate whose derivative is computed in accordance with Eqs. (3.2). The parentheses denote scalar products.

In accordance with the maximum principle [2], we introduce the vectors of the conjugate variables p_x and p_y associated with the vectors x and y, and construct the Hamiltonian, conjugate equations, and transversality conditions for problem (3.2), (3.3),

$$H = (p_x, y) + (p_y, v) + 2(x, y), \qquad dp_x / dt = -2y$$
$$dp_y / dt = -p_x - 2x, \qquad p_x(1) = p_y(1) = 0 \tag{3.4}$$

As usual, we assume that $p_0 \equiv -1$. From equations and boundary conditions (3.2), (3.4) we obtain successively

$$dp_x / dt = -2dx / dt, \qquad p_x(t) = 2[x(1) - x(t)]$$
$$dp_y / dt = -2x(1), \qquad p_y(t) = 2. (1)(1 - t) \tag{3.5}$$

By virtue of the condition of maximality of the function H with respect to v, the vector v is colinear with the vector p_y and equal to its maximal value $|v| = 1$ in magnitude. Since (by (3.5)) the vector p_y is of constant direction, it follows that $v(t) = f$ for $\tau \leq t \leq 1$, where f is a constant unit vector. Integrating system (3.2) under initial conditions (3.3), we obtain

$$x(1) = x^+ + y^+(1 - \tau) + f(1 - \tau)^2 / 2 \tag{3.6}$$

We choose the vector f on the basis of the maximality condition for the expression for $x^2(1)$ as defined by Eq. (3.6). Maximizing, we obtain

$$f = [x^+ + y^+(1 - \tau)] / |x^+ + y^+(1 - \tau)|$$
$$|x(1)| = |x^+ + y^+(1 - \tau)| + (1 - \tau)^2 / 2 \tag{3.7}$$

We set $x^+ = x^-$, $y^+ = y^- + U$ in relation (3.6), where $x^- = x(\tau - 0)$, $y^- = y(\tau - 0)$ are the values of the coordinates and velocities immediately prior to correction. This yields

$$|x(1)| = (1 - \tau)\,|s + U| + (1 - \tau)^2 / 2, \quad s = y^- + x^- / (1 - \tau) \tag{3.8}$$

Let us find the minimum of the expression for $|x(1)|$ from (3.8) on the basis of the vector $|U| \leq k$ in accordance with (3.1). It is easy to show that the minimizing vector U and the minimal value of $|x(1)|$ are given by

$$U = -s; \qquad |x(1)| = (1 - \tau)^2 / 2 \quad \text{for } |s| \leqslant k$$
$$U = -ks / |s|, \qquad |x(1)| = (1 - \tau)(|s| - k) + (1 - \tau)^2 / 2 \quad \text{for } |s| \geqslant k \tag{3.9}$$

From Eqs. (3.9) we see that $|x(1)|$ depends monotonously on $|s|$. To compute the first maximum in (3.1) it is sufficient to solve the problem of optimal control of system (3.2) in the interval $[0, \tau]$ with zero initial conditions (1.3) and maximizing functional $|s|$. The solution of this problem can be obtained by taking account of the self-evident inequality

$$\max |s| \leqslant \max |y(\tau - 0)| + (1 - \tau)^{-1} \max |x(\tau - 0)|$$

Clearly, each individual maximum in the right side of this inequality is attained if the control v in system (3.2) with zero initial conditions (1.3) is taken to be of maximal magnitude and arbitrary but constant in direction. Hence, the maximum of $|s|$ is attained if we take $v = e$ for $0 \leq t \leq \tau$, where e is an arbitrary constant vector of unit length. Integrating system (3.2) and computing s from Formula (3.8) and $|x(1)|$ from Formulas (3.9), we obtain

$$y^- = e\tau, \qquad x^- = e\tau^2 / 2, \qquad |s| = \tau + \tau^2 [2(1 - \tau)]^{-1} \tag{3.10}$$
$$|x(1)| = (1 - \tau)^2 / 2 \quad \text{for } |s| \leqslant k, \qquad |x(1)| = 1/2 - k(1 - \tau) \quad \text{for } |s| \geqslant k$$

It is easy to show that $|s|$ depends monotonously on τ, and that the minimum of $|x(1)|$ with respect to τ from the interval [0, 1] is attained when $|s| = k$. From this condition and relations (3.10) we obtain the optimal correction instant τ_2 and functional (3.1) for the case under consideration,

$$\tau_2 = 1 + k - \sqrt{1 + k^2}, \qquad J_2 = (1 - \tau_2)^2 / 2 = (1 + 2k^2 - 2k\sqrt{1 + k^2}) / 2 \qquad (3.11)$$

4. Continuous correction of a pulse disturbance. The control and disturbance are restricted as follows: $|u(t)| \leq k$ for $0 \leq t \leq 1$ and $v = V\delta(t - \theta)$, where $|V| \leq 1$, $0 \leq \theta \leq 1$. It is clear that correction must not be made prior to the action of the disturbance, i.e. that $u = 0$ for $0 \leq t \leq \theta$. It is also clear that we must proceed on the basis of a maximal disturbance pulse, i.e. $V = e$, where e is a unit vector of arbitrary direction. The motion of the system following the action of the disturbance, i.e. for $t \geq \theta$, is described by the equations and initial conditions

$$dx / dt = y, \quad dy / dt = u, \quad |u(t)| \leqslant k, \quad x(\theta) = 0, \quad y(\theta) = e \qquad (4.1)$$

It is clear that optimal correction requires that the control $u(t)$ be directed opposite to the disturbance pulse and that it be of the maximal magnitude k, i.e. that $u = -ke$, if the deviation of $|x(1)|$ at the end of the process cannot be reduced to zero. Hence, integrating Eqs. (4.1), we obtain

$$|x(1)| = (1 - \theta) - k(1 - \theta)^2 / 2 \quad \text{for} \quad 1 - \theta - k(1 - \theta)^2 / 2 \geqslant 0$$
$$|x(1)| = 0 \quad \text{for} \quad 1 - \theta - k(1 - \theta)^2 / 2 \leqslant 0 \qquad (4.2)$$

The latter case corresponds to the situation where the disturbance can be neutralized completely. Determining the maximum of Expression (4.2) for $|x(1)|$ with respect to θ from the interval [0, 1], we obtain the disturbance instant θ_3 least favorable from the correction standpoint and the associated minimax value of the functional $J_3 = |x(1)|$ in the case under consideration,

$$\theta_3 = 0, \quad J_3 = 1 - k/2 \quad \text{for} \quad 0 \leqslant k \leqslant 1$$
$$\theta_3 = 1 - 1/k, \quad J_3 = 1/2k \quad \text{for} \quad k \geqslant 1 \qquad (4.3)$$

5. Pulse correction of a pulse disturnance. We set $u = U\delta(t - \tau)$, $v = V\delta(t - \theta)$, where $|U| \leq k$, $|V| \leq 1$, $0 \leq \tau$, $\theta \leq 1$. If $k \geq 1$, then the disturbance can be completely neutralized with any degree of accuracy by setting $U = -V$, $\tau = \theta$. In order to obtain the minimax solution in the case $k < 1$ we must clearly set $V = e$, $U = -ke$, where e is an arbitrary unit vector (as above). According to Eqs. (1.3), the deviation at the end of the process is

$$|x(1)| = 1 - \theta - k(1 - \tau)$$

First let us find the minimum of this quantity with respect to τ for $\theta \leq \tau \leq 1$ (correction is effected only after the direction of the disturbance is determined), and then its maximum with respect to θ for $0 \leq \theta \leq 1$. We then obtain the optimal instants of disturbance and correction and the value of the functional,

$$\tau_4 = \theta_4 = 0, \quad J_4 = 1 - k \quad \text{for} \quad 0 \leqslant k \leqslant 1, \qquad J_4 = 0 \quad \text{for} \quad k \geqslant 1 \qquad (5.1)$$

Thus, in this case (in contrast to the two preceding ones) both the disturbance and the correction are best effected at the beginning of the process.

6. Discussion of the results. Comparing Eqs. (2.1) and (3.11), we can see that $J_1 \geq J_2$ for $0 \leq k \leq 3/4$ and $J_1 \leq J_2$ for $k \geq 3/4$. Hence, if the disturbance is continuous (bounded in absolute value), then it is more expedient to employ pulse correction for $k \leq 3/4$ and continuous correction for $k \geq 3/4$. In other words, if the total correction force pulse is sufficiently large ($k \geq 3/4$), then it is advisable to expend it gradually, effecting more precise continuous correction of the continuous disturbance. If the total correction pulse is small, it is better to concentrate it in a single pulse. Relations (4.3) and (5.1) imply that $J_3 \geq J_4$ for all k, i.e. that in the case of a pulse disturbance pulse correction is always more expedient than continuous correction. We note, moreover, that $J_i\,(k) = 0$ only for $i = 1$, $i = 4$ and $k \geq 1$, i.e. that exact correction is possible in those and only in those cases where it is of the same character as the (continuous or pulse) disturbance, and when the total pulse is sufficiently large. The Fig. 1 shows the functions $J_i\,(k)$ as given by relations (2.1), (3.11), (4.3), and (5.1), where the number $i = 1$, 2, 3, 4 is indicated next to each curve.

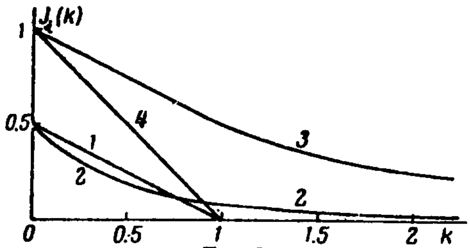

Fig. 1

7. Solution without restrictions.

In conclusion, let us solve the problem of the minimax of the functional $J = |x(1)|$ for system (1.3) under integral restrictions (1.3) only, i.e. without any of the additional restrictions imposed in Sections 2 to 5. If $k \geq 1$, then, as in Sections 2 and 5, we can choose our control in the form $u(t) = -v(t)$ in the interval $0 \leq t \leq 1.$, so that in this case $J = 0$. If $k < 1$, then the minimax solution requires that the disturbance and control be of constant and opposite direction, i.e. that $v = |v|\,e$, $u = -|u|\,e$, where e is a constant unit vector of arbitrary direction. Then, integrating Eqs. (1.3), we obtain

$$|x(1)| = \int_0^1 (1 - t)\,[\,|v(t)| - |u(t)|\,]\,dt$$

This relation and integral restrictions (1.3) imply that the minimax of the functional $|x(1)|$ is attained with a pulse disturbance and pulse correction, with both pulses applied at the beginning of the process. Thus, we have $v = e\,\delta(t)$, $u = -ke\,\delta(t)$. The resulting solution soincides fully with that of Section 5 for the case of pulse correction of a pulse disturbance. In particular, $J = J_4$, where J_4 is given by relation (5.1).

BIBLIOGRAPHY

1. Isaacs, R., Differential Games. "Mir" Press, Moscow, 1967.
2. Pontriagin, L.S., Boltianskii, V.G., Gamkrelidze, R.V. and Mishchenko, E.F., The Mathematical Theory of Optimal Processes. Fizmatgiz, Moscow, 1961.

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